

A NEW TOPOLOGICAL PERSPECTIVE ON QUANTIZATION IN PHYSICS

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ABSTRACT. We propose a new characterization of classical quantization in physics in terms of sheaf cohomology on the site of spacetime as a smooth 4-manifold. The perspective of sheaf cohomology is motivated by a presentation of the Aharonov-Bohm effect in terms of the integration of differential forms.

1. INTRODUCTION

Awkward inconsistencies have notoriously plagued quantum mechanics almost from the moment of its birth, and certainly ever since the Copenhagen interpretation crystallized into a prevailing dogma, its *ad hoc* non-classical statistics notwithstanding. A noteworthy example of this lamentable state of affairs is the interpretational status of the Aharonov-Bohm experiment dating back to the early 1960's. R. Doll and M. Nábauer [8] and, separately, W. Fairbank and B. Deaver [6] provided experimental verification of the existence of a smallest amount of flux in the sense of a quantization unit of flux in the amount of $\frac{h}{2e}$. Prior to this, however, in the context of their original experimental work of 1959, Y. Aharonov and D. Bohm [1] produced a value of $\frac{h}{e}$ for this minimum unit: the Aharonov-Bohm law. This lack of agreement reveals the presence on the scene of an unsatisfactory foundation for interpretation of the indicated experimental data centered on nothing less than the question of whether and how certain mainstream quantum mechanical statistical methods should be brought to bear on the situation.

Much of this interpretational difficulty is centered on the fundamental question of whether, and, if so, to what degree, we are dealing with a disordered quantum mechanical system. Taking this one step further, ultimately the sticking point is concerned with the epistemological dichotomy that pits a single system quantum mechanical reality against that of an ensemble of many particles. Indeed, we now find ourselves face to face with the mathematically and philosophically unacceptable manoeuvre of requiring an individual particle (or single quantum mechanical system) to manifest the behavior of an ensemble, as though each member of a set must be perfectly average.

Doubtless it is folly, however, to embark on a philosophical critique or criticism of this conceit of quantum mechanics at this point in history simply because, happily,

Received by the editors August 2 (ff.), 2011.

2000 *Mathematics Subject Classification*. Primary 05C38, 15A15; Secondary 05A15, 15A18.

Key words and phrases. Quantization, de Rham cohomology, sheaf cohomology.

This paper is in final form and no version of it will be submitted for publication elsewhere.

the mathematical formalisms of quantum mechanics are so well established that to attempt to find fault with this machinery is absurd: there is no question that we are dealing with a smashingly successful theory. Additionally, if we take into account the beautiful interplay between the so-called Schrödinger and Heisenberg pictures, bringing the Dirac picture into play when relativistic effects are significant, and add to the mix the overarching functional analytic framework built on the theory of densely defined Hermitian (unitary) operators on Hilbert spaces of states, what we have before us is unarguably an elegant and well-developed mathematical theory geared toward quantum mechanical questions.

Going in the other direction, however, what if we take this mathematical edifice as something of a starting point in itself and ask whether there are mathematical constructs available that were at the same time consonant with the foregoing functional analysis and amenable to an approach to quantum mechanical interpretational questions (such as the aforementioned Aharonov-Bohm affair) that would skirt *ad hoc* suppositions like those that mar the Copenhagen interpretation. Additionally, with the explicit example of the Aharonov-Bohm experiment to guide us, we suggest at the outset that such a new approach should include the mathematical perspectives afforded by algebraic topology and differential geometry in its modern manifestations.

Indeed, in the corresponding broader context in which the indicated physical questions can be phrased, that of electrodynamics, we can start with a revisit to Maxwell's equations, noting that these can be formulated in a wonderfully economical integral formulation [14]. We discussed this at some length in [19] which constitutes something of a precursor to what we do in what follows (which is however self-contained). For our present purposes all we need to do is observe that Maxwell's equations can be readily rendered as statements about integrals of differential 2- and 3-forms associated with electric and magnetic fields defined locally on spacetime as a smooth 4-manifold (cf. loc. cit., section 2.1)

Here then, cutting to the chase, is the lynchpin of what we are attempting to do: the quantum mechanics and local electrodynamics discussed in the foregoing, i.e., nothing less than the behavior of quantization and the rendering of quantum mechanical processes along the lines of what takes place with the Aharonov-Bohm experiment, should be regarded and explicated in terms of a cohomology theory. With the 4-manifold of spacetime as the ambient topological space we are quickly directed to de Rham cohomology; however, for mathematical reasons that we make clear below, it is actually sheaf cohomology that will soon take the lead role.

Even as the Aharonov-Bohm experiment and the inconsistencies in its explication as related above provide us with physical rationale for what follows, the ultimate scope of what we propose as regards new topological and differential geometric perspectives (and methodologies) is far broader and more ambitious. Specifically it is the case that the sheaf cohomology we bring to bear on the algebra of smooth differential forms on spacetime (*viz.* a so-called de Rham complex) makes it possible for us to define quantization itself in a very natural mathematical way, immediately opening the door to a new characterization of phenomena within the purview of quantum mechanics, i.e. measurables and observables, in terms of relations between sheaves, and therefore between their fibres and stalks.

So it is, then, to boot, that given that sheaves are the mathematical tool *par excellence* for addressing the question of local *vs.* global data and information,

what we propose here may contain the germ of a novel vision of the connection between what Dirac refers to as physics "in the small" and physics "in the large," in his unsurpassed monograph [7].

Finally, before we get to the business at hand, the authors wish to express their gratitude to Jan Post, and over his strenuous objections, which are hereby overruled, stress that the ideas behind these exciting developments are entirely his. He is the main player in this game.

2. BACKGROUND FROM PHYSICS: CASTING CERTAIN PHYSICAL LAWS IN TERMS OF DIFFERENTIAL FORMS

In [18] and [19] we developed a detailed formalism for a certain class of physical laws from the areas of quantum mechanics and (classical) electrodynamics expressed in the language of differential forms on spacetime as a smooth 4-manifold. We refer the reader to these articles by way of a prelude to what follows. However, as we already indicated earlier, our goal is to make the present discussion as self contained as possible (certainly as far as the mathematics goes), so we reiterate a number of items from this earlier article.

If M stands for the Riemannian 4-manifold of spacetime, following standard procedure in differential geometry, write $\Omega^p(M)$ for the linear space of differential p -forms (of class C^∞) on M . Thus a typical element of $\Omega^p(M)$ looks like $\omega = \sum_I a_I dx_I$, with I a p -fold multi-index as usual, and each a_I an infinitely differentiable function on M . The examples we start off with in this section are 2- and 3-forms that have very fundamental physical interpretations.

Write, as usual, \mathbb{E} , \mathbb{B} , \mathbb{D} , \mathbb{H} , and \mathbb{J} for, respectively, the intensity of the electric field, the density of the magnetic field, flux density, the intensity of the magnetic field, and electric current density. Additionally, \mathbb{A} is the magnetic field potential, φ is the electric potential, and ϱ stands for the free charge density. Then, as is well known, Maxwell's equations can be given as

$$(2.1) \quad \mathbb{E} = -\nabla\varphi - \mathbb{A}_t$$

$$(2.2) \quad \mathbb{B} = \nabla \times \mathbb{A}$$

$$(2.3) \quad \nabla \cdot \mathbb{D} = \varrho$$

$$(2.4) \quad \nabla \times \mathbb{H} - \mathbb{D}_t = \mathbb{J}$$

Following Minkowski [16], the indicated magnetic and electric properties of the field can be joined into the components of spacetime tensors in such a way that \mathbb{E} and \mathbb{B} are joined together into a tensor F_{lk} while \mathbb{H} and \mathbb{D} get joined into G^{lk} , with the indices l, k ranging over the values 0, 1, 2, 3, with 0 the time label and 1,2,3 the space labels. Then the preceding quartet is replaced by the equations

$$(2.5) \quad \partial_s F_{kl} + \partial_k F_{sl} + \partial_l F_{ks} = 0$$

$$(2.6) \quad \partial_l A_k - \partial_k A_l = 0$$

$$(2.7) \quad \partial_k G^{lk} = C^l$$

$$(2.8) \quad \partial_k C^k = 0$$

all with the usual tensor calculus conventions in place (including Einstein summation). Here the first relation, a generalized curl, captures a local Faraday-Maxwell law; the second relation defines vector potential (A); the third relation, a generalized divergence, is nothing else than Ampère's law for displacement current; and the fourth relation expresses local charge conservation.

After this, Einstein, bringing the perspective of general relativity to bear on the situation, replaced Minkowski's partial derivatives by covariant derivatives. Under this scheme (2.5) and (2.6) stay the same in appearance (even though the operators' meaning has changed) because of the cancellation of the Christoffel terms. However, (2.7) and (2.8) each acquire an extra term involving Christoffel symbols. The structural effect of this move is to expose a natural symmetry or data pairing in the sense that (2.5), (2.6) evince what we will call the Faraday data in Maxwell's formalism, whereas (2.7), (2.8) constitute what we will call the Ampère data; see [19] for further discussion. It was the Austrian physicist Friedrich Kottler [15] who observed that in this arrangement the contravariant objects G^{lk} and C^k should be interpreted as tensor and vector densities of weight -1 , and it was Elie Cartan who then realized that this insight on Kottler's part is amenable to a phrasing in terms of the geometry and analysis of differential forms, if one starts with Maxwell's equations in their equivalent form as integral equations. The upshot is that we then obtain, in addition to Gauß' law, what we will call the Maxwell-Faraday data

$$(2.9) \quad \int_{\partial S} \mathbb{E} \cdot dl = -\frac{\partial \varphi_{\mathbb{E},S}}{\partial t}$$

and the Maxwell-Ampère data

$$(2.10) \quad \int_{\partial S} \mathbb{B} \cdot dl = \mu_0 I_S + \mu_0 \frac{\partial \varphi_{\mathbb{E},S}}{\partial t}$$

where $\varphi_{\mathbb{B},S}$ and $\varphi_{\mathbb{E},S}$ represent, respectively, magnetic and electric flux through the surface S .

There are now three observations to be made regarding the content of (2.9) and (2.10). First, Kottler's interpretation indicates a physically meaningful duality for the surface S . In other words, what we have here is generalized curl and divergence acting as essentially dual operations. Second, from a strategic point of view we should take note of the fact that we are positing Maxwell's equations in a metric-free environment: it is all a matter of topology. And, third, the integrals are indeed integrals of differential forms on M , effected over the boundary of an immersed surface S . It is the latter observation that points us in the direction of de Rham cohomology.

To emphasize this third point even more emphatically, suppose that M is endowed with local coordinates x_i , $1 \leq i \leq 3$, and (for time) t . If we encode the Gauß-Ampère data of Maxwell's formalism in terms of the earlier tensor (equipped with an obvious new notation) \tilde{C} and an evident companion tensor $\tilde{G} = \tilde{G}(M, \mathbb{D})$, we can write, e.g.,

$$(2.11) \quad \iint_S \tilde{C}_{\lambda\nu\kappa} dx_1^\lambda dx_2^\nu dx_3^\kappa = \iint_{\partial S} \tilde{G}_{\lambda\nu} dx_1^\lambda dx_2^\nu$$

where $\tilde{G}_{\lambda\nu} dx_1^\lambda dx_2^\nu \in \Omega^2(M)$ and $\tilde{C}_{\lambda\nu\kappa} dx_1^\lambda dx_2^\nu dx_3^\kappa \in \Omega^3(M)$, so that we recognize in (2.9) an instance of Stokes' theorem in its most general form [2] (also [21]):

$$(2.12) \quad \int_S d\omega = \int_{\partial S} \omega$$

where d represents the exterior derivative and ∂ represents the (homological) boundary operator. It is a commonplace of differential geometry that this lynchpin of calculus on manifolds sits at the heart of de Rham cohomology.

However, before we get to that, it is appropriate to add a little more critical analysis of our motivating quantum mechanical phenomena, i.e., the aforementioned Aharonov-Bohm effect. We stipulate at this stage that, to be sure, the flux quantization read from the experiment of 1959 discussed in the previous section should come in multiples of the single unit $\frac{h}{2e}$, supporting both [8] and [6]; in addition to these sources the reader should also consult our [19], as well as Post's [17]. We also demonstrated in [19] that if we start with Gauß-Ampère data phrased in terms of differential forms by requiring that global flux conservation be presented as

$$(2.13) \quad \iint_c F = 0$$

for all 2-cycles c and with F (flux) an exact form, then we can retrieve the attendant potential field as a 1-form A , and, most importantly, we obtain the fundamental relation

$$(2.14) \quad \int_{\gamma} A = n \frac{h}{2e}$$

for all 1-cycles γ , with n being the number of \pm -linked elementary flux units linked by γ .

What we encounter here in altogether explicit terms is the emergence of nothing less than the quantization of flux, as demonstrated experimentally by Aharonov and Bohm in 1959 and explicated by Fairbank and Deaver in [6]; and the present characterization of flux is entirely a result of intrinsic topological properties of the potential field $A \in \Omega^1(M)$: the requisite integer multiples of $\frac{h}{2e}$ arise as periods of A .

Now for quantization *via* de Rham (or, rather, sheaf) cohomology.

3. QUANTUM MECHANICS, DE RHAM AND SHEAF COHOMOLOGY, AND SPACETIME

In a number of previous publications, i.e. [17], [18], [19], we have put forth the thesis that in light of such considerations as our discussion of the Aharonov-Bohm experiment, and a lot more besides, the *ad hoc* and non-classical statistical methods introduced into quantum mechanics by Born and promoted by Bohr and his followers, creating the dogma of the Copenhagen interpretation, should be critically revisited and amended. In [18], and already in certain places in [17], it was argued that a two-tier approach to quantum mechanics should be brought to bear upon quantum reality; statistical methods naturally apply to ensembles, or quantum systems of many particles, but for a single system, i.e. a single particle, we propose a methodology rooted in differential geometry and algebraic topology. What we do in the present paper is the next step in this program, namely, the delineation of quantization itself by means of the indicated cohomological formalism.

It is this complementarity of methodologies, contrasting disordered quantum mechanical systems with ordered ones, that we suggest as a general epistemological principle *vis à vis* the physical analysis of the microscopic domain. As we shall demonstrate in a later publication, our approach through sheaf (and de Rham) cohomology also makes for a number of deep connections with physics in the macroscopic domain because what we do is in many fundamental ways independent of the (Lorentz) metric space structure placed on spacetime by Einstein's relativity. But we stress at this point in the proceedings that our set of proposals most directly pertain to the physics of the microscopic domain where they both correct long-standing philosophical anomalies in the Copenhagen interpretation, and provide uniform mathematical tools with which to explicate certain classes of (ordered) quantum mechanical phenomena, the fractional quantum Hall effect being a prime example.

As we pointed out at the close of section 2, the lynchpin of our formalism is the integration of differential forms over cycles, and we now come to the heart of the matter, indeed the heart of quantum mechanics itself, *viz.* quantization. Our formalism must include a mechanism whereby to "tag" admissible output, i.e., output in the shape of sets of integer multiples of a single unit. The motivating example of the Aharonov-Bohm effect asks for integer multiples of the quantum unit $\frac{h}{2e}$, and we showed in [19] that period integrals do the job; in the present

context see (2.14). This is what we now generalize and couch in the language of de Rham and sheaf cohomology.

Given that what we do is heavily steeped in the formalism of cohomology in the style of Alexander Grothendieck's revolutionary work in algebraic geometry dating to the 1960s and 1970s, we present a primer of sorts on this material in the Appendix to this paper. Our present objective is to lay out our novel geometric characterization of quantization in as concise a fashion as possible. Mathematically speaking the i's are dotted and the t's are crossed in the aforementioned Appendix.

Our physical analysis now proceeds in two parts. First, in the remainder of the present section we quickly delineate the axioms engendering quantization; then, second, we address the question of physical interpretation of this mathematical formalism, and how it relates to certain themes in mainstream quantum mechanics, in the next section.

The fundamental idea driving the discussion is that, with (2.14), for instance, as a paradigm, there should be available an algebraic topological means whereby to require that only output data in the form of finite subsets of the \mathbb{Z} -module

$$(3.1) \quad \hbar\mathbb{Z} := \{n\hbar\}_{n \in \mathbb{Z}},$$

where we have chosen Planck's constant \hbar principally for convenience (see below), should be admissible. In other words we need a condition (or axiom) ensuring that our period integrals will single out data belonging to $\hbar\mathbb{Z}$, the observable or measurable output of the corresponding quantum mechanical experiments. With sheaf cohomology in place, and also Hodge theory, this turns out to be structurally quite simple. To wit:

With M being spacetime, so that $0 \leq p \leq 4 = \dim(M)$, and with the object \mathcal{J}^p defined below (cf. (3.4)), define the relation $\equiv (\text{mod } \mathcal{J}^p)$ as follows: for $\omega, \tau \in \Omega^p(M)$,

$$(3.2) \quad \omega \equiv \tau (\text{mod } \mathcal{J}^p)$$

if and only if, for all p -chains γ on M ,

$$(3.3) \quad \int_{\gamma} (\omega - \tau) \in \hbar\mathbb{Z}.$$

It is a triviality to check that (3.2) and (3.3) define a relation on $\Omega^p(M)$ which is reflexive, symmetric, and transitive; in other words, we have an equivalence relation on each $\Omega^p(M)$ and it is immediate that this equivalence factors through to $\Omega^p(M)^\dagger$, the sheaf of germs of smooth p -forms on M (see the Appendix for specifics about such sheaves of germs).

Therefore, by the usual convention regarding equivalence relations of this type, \mathcal{J}^p can be identified with the "0-class"

$$(3.4) \quad \mathcal{J}^p = \{\omega \in \Omega^p(M) \mid \int_{\gamma} \omega \in \hbar\mathbb{Z}, \text{ for all } p\text{-chains } \gamma\}$$

and we get, too, in the obvious notation $\mathcal{J}^{p\dagger}$ for the image of \mathcal{J}^p in $\Omega^p(M)^\dagger$. Obviously \mathcal{J}^p is a \mathbb{Z} -submodule of $\Omega^p(M)$.

Quantization is therefore part and parcel of requiring that the differential forms that can play are elements of \mathcal{J}^p for some p . This is clearly an intrinsic characterization, so the next task before is to address the issue of how to get at this restriction extrinsically; we make a start to this discussion in the next section.

With apologies for the concomitant increase in abstraction (and accordingly again referring to the Appendix), we close the present section with a few observations regarding the cohomological specifics accruing to stipulations we just made. First off, we are of course in a position to write down the following short exact sequence

$$(3.5) \quad 0 \rightarrow \mathcal{J}^{p\dagger} \xrightarrow{\iota} \Omega^p(M)^\dagger \rightarrow \text{coker}(\iota) \rightarrow 0$$

which we regard as living in the abelian category of sheaves over spacetime. Here $\text{coker}(\iota)$ is really nothing else than the quotient sheaf $\Omega^p(M)^\dagger/\mathcal{J}^{p\dagger}$ (whose construction may actually require a little extra, namely, the process of sheafification; see [12], for instance). This in turn yields another fine resolution

$$(3.6) \quad 0 \rightarrow \Omega^0(M)^\dagger/\mathcal{J}^{0\dagger} \xrightarrow{\eta} \Omega^\bullet(M)$$

where η is the natural map on the set of representatives of the earlier equivalence relation acting on $\Omega^0(M)^\dagger$. (This definition may actually involve the Axiom of Choice.) Bringing in the machinery of sheaf cohomology, there are actually two cohomology theories (or cohomology functors) that can be ascribed to the geometrical data $\Omega^\bullet(M)$ with an eye toward quantization. To wit, we have

$$(3.7) \quad H^\bullet(M, \mathcal{J}^{0\dagger}) = R^\bullet(M, \mathcal{J}^{0\dagger})$$

by virtue of the obvious fine resolution

$$(3.8) \quad 0 \rightarrow \mathcal{J}^{0\dagger} \rightarrow \Omega^\bullet(M)^\dagger,$$

and we have

$$(3.9) \quad H^\bullet(M, \Omega^0(M)^\dagger/\mathcal{J}^{0\dagger}) = R^\bullet\Gamma(M, \Omega^0(M)^\dagger/\mathcal{J}^{0\dagger})$$

in view of (3.7).

So this provides the first part of the scaffolding for a full treatment of quantization along the indicated novel geometrical lines.

4. PHYSICAL ASPECTS OF OUR APPROACH TO QUANTIZATION

The crux of our scheme for quantization *via* an equivalence relation (i.e. (3.2) and (3.3)) on suitable differential forms is the stipulation that for any given p , the ideal, or \mathbb{Z} -module, \mathcal{J}^p provides us with the differential forms that have meaning in quantum mechanical processes. This immediately raises the obvious question of why this should be so. This question is clearly central to our entire enterprise of quantum mechanics: why should nature's laws be structured in such a way that phrasing them in the present language of period integrals reveals that in order to have quantization (philosophically speaking an empirically verified reality), an *a priori* selection procedure appears to be in place, singling out the elements of \mathcal{J}^p ?

One way of looking at this situation, admittedly embodying only an embryonic idea at this stage, is to note that the form of the equivalence relation we propose naturally provides us with what we might characterize as a moduli space. In other words, as per (3.6), the quotient sheaf $\Omega^0(M)^\dagger/\mathcal{J}^{0\dagger}$, and more generally the objects $\Omega^p(M)^\dagger/\mathcal{J}^{p\dagger}$, provide a natural mathematical construct for carrying out physical, or quantum mechanical, procedures and calculations replete with the guarantee that the output will be quantized. Thus, by positing that we are actually working in a moduli space setting we gain the immediate advantage of having quantization brought out without *ad hoc* stipulations that seek to jettison various notions from, for instance, classical mechanics, ultimately making for a dissonance between classical and quantum physics that flies in the face of all proper epistemology.

To be sure, there is a lot still hiding in the shadows as far as the ramifications of our proposals regarding quantization are concerned, but there is a great advantage to be gained already simply by stressing that our approach centers on geometry: quantum mechanics as such "lives" on a moduli space. Thus, there is no question of our asking for, say, non-classical statistics in the sense of the Copenhagen interpretation of quantum mechanics, ascribing to a single particle individually, i.e. to a single quantum mechanical system, the data coming from an ensemble of such particles, as though an average trait must be descriptive of every element of a sample space. There does remain a separate issue to be discussed in our scheme, namely, the transition between the macroscopic and microscopic domains of discourse as exemplified by the fact that classical mechanics applies at larger scales (and non-relativistic speeds), whereas things are very different at smaller (simultaneously relativistic) scales. But we propose that this seemingly anomalous state of affairs should be treated geometrically in the sense we employ above.

It is apposite to note in this connection that our earlier paradigm of the Aharonov-Bohm experiment certainly points in this direction in no uncertain terms. This particular theme was explored in some detail already in [18], where the proposal was put forth that in the microscopic domain there should be, so to speak, a two-tiered reality in place. By means of analyzing the Aharonov-Bohm experiment it was argued there that a more conservative (and mathematically reasonable) approach to the according quantum mechanical phenomena would be forthcoming by stipulating that a path integral formalism (of the sort we start with in the present paper) not only replaces certain problematical approaches in quantum mechanics done according to the Copenhagen recipe, but it opens up a more holistic way of approaching the whole quantum mechanical enterprise.

We pursue the latter theme in two upcoming publications, one presenting a careful historical-critical dissection of a number of themes comprising part of the

Copenhagen interpretation and placing our approach to quantization by means of sheaf cohomology in a proper historical framework, the other addressing, among other things, broader connections with themes belonging to the geometry of space-time and attendant symmetry considerations. These, and especially the latter, will hopefully serve to bolster our emerging thesis that differential geometry and sheaf cohomology centered on the ideas of de Rham point toward a more organic presentation of quantum mechanics and open the door to connecting it smoothly to other parts of physics, especially general relativity, all without doing violence to either mathematical or philosophical sensibilities.

5. APPENDIX. BACKGROUND FROM ALGEBRAIC TOPOLOGY: MANIFOLDS, DE RHAM COHOMOLOGY, SHEAF COHOMOLOGY

As promised, in this section we collect mainly algebraic topological and algebraic geometric material relevant to the preceding. Most, if not all, of the differential geometric material underlying it all is standard and can be found in any number of textbooks on the subject, e.g. Bott's and Tu's beautifully written book [2]. However, for the specific business of making the transition from de Rham cohomology to sheaf cohomology we refer the reader to the (now unfortunately rather rare) text [5] by Chern. This latter source also contains a very readable and compact discussion of Grothendieck's proof [11] of de Rham's theorem, which is at the heart of the transition just mentioned. Speaking of Grothendieck, for the details of sheaf cohomology, his classic SGA series [10] is still definitive, but more accessible treatments are readily available; see e.g. [3] and [22] for general themes and the gorgeous work [13], for example, for computation of sheaf cohomology by resolutions.

As we mentioned before, for our purposes the ambient space is a finite dimensional smooth manifold; we note that the class of smooth manifolds is of course a subclass of the class of topological spaces. So, if M is (now not just spacetime but) any such smooth manifold of dimension $n < \infty$, then we have, immediately, that locally (i.e., at and in a neighborhood of every $x \in M$) there are tangent space structures available: for every $x \in M$ there is an open neighborhood of x , say, U_x , in M , and a diffeomorphism from U_x into \mathbb{R}^n such that if we have local coordinates (x^1, \dots, x^n) in U_x , then the set $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ yields a basis for the tangent space $T_x(M)$ at x ; in other words, $T_x(M) = \bigoplus_{\nu=1}^n \mathbb{R} \frac{\partial}{\partial x^\nu}$. Under these circumstances the cotangent space at x is the dual space $Hom_{\mathbb{R}}(T_x(M), \mathbb{R}) = T_x^*(M) = \bigoplus_{\nu=1}^n \mathbb{R} dx^\nu$, spanned by the first differentials of the local coordinates. Thus, if \langle, \rangle denotes the standard dual vector space pairing, we have the fundamental relationship $\langle \frac{\partial}{\partial x^\mu}, dx^\nu \rangle = \delta_{\mu\nu}$ (Kronecker's delta) in place. The preceding local data is rendered global simply by defining $T(M) = \cup_x T_x(M)$ and $T^*(M) = \cup_x T_x^*(M)$.

The cotangent bundle $T^*(M)$ is dealt an exterior product structure *via* the wedge product by defining

$$(5.1) \quad dx^\nu \wedge dx^\mu = -dx^\mu \wedge dx^\nu$$

$$(5.2) \quad dx^\nu \wedge dx^\nu = 0$$

Consequently we can iteratively build expressions of the form $dx^{\nu_1} \wedge dx^{\nu_2} \wedge \dots \wedge dx^{\nu_p}$ subject to the rules, first, that if σ is any permutation of $1, 2, \dots, p$, then

$$(5.3) \quad dx^{\nu_{\sigma(1)}} \wedge dx^{\nu_{\sigma(2)}} \wedge \dots \wedge dx^{\nu_{\sigma(p)}} = (-1)^{\text{sgn}(\sigma)} dx^{\nu_1} \wedge dx^{\nu_2} \wedge \dots \wedge dx^{\nu_p}$$

and, second, if, for some $i \neq j$ we have $\nu_i \neq \nu_j$, then

$$(5.4) \quad dx^{\nu_1} \wedge dx^{\nu_2} \wedge \dots \wedge dx^{\nu_p} = 0$$

The class of p -forms on M , written $\Omega^p(M)$, as we have already seen above, obtains as the free abelian group on the set of all \mathbb{R} -linear expressions generated by the objects $f(x^1, x^2, \dots, x^p) dx^{\nu_1} \wedge dx^{\nu_2} \wedge \dots \wedge dx^{\nu_p}$ with each f of class C^∞ . Thus, a typical element of $\Omega^p(M)$ looks like

$$(5.5) \quad \omega = \sum_{1 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_p \leq n} f_{\nu_1 \nu_2 \dots \nu_p}(x^1, x^2, \dots, x^p) dx^{\nu_1} \wedge dx^{\nu_2} \wedge \dots \wedge dx^{\nu_p}$$

If, for convenience, we just write I for the ordered multi-index $1 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_p \leq n$, then we get the obvious notational simplification

$$(5.6) \quad \omega = \sum_I f_I dx^I$$

The exterior derivative d possesses one more property, one that defines it as a differential and a coboundary operator:

$$(5.7) \quad d \circ d = d^2 = 0$$

It is this property that gives rise to de Rham cohomology. Specifically, seeing that d must obey the rule

$$(5.8) \quad d\omega = d(\sum_I f_I dx^I) = \sum_I df_I \wedge dx^I$$

and seeing that each df_I is a total differential in the sense of elementary calculus, a trivial calculation shows that $d\omega$ must be a $(p+1)$ -form; in other words, we have, for all $p \geq 0$,

$$(5.9) \quad d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$$

But now, using (5.7), we obtain from (5.9) that we have a cochain complex

$$(5.10) \quad \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-2}(M) \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M)$$

denoted $\Omega^\bullet(M)$, for which we have that $\Omega^0(M) = C^\infty(M)$, the map $\Omega^0(M) \xrightarrow{d} \Omega^1(M)$ is injective (i.e., one-one), and $\Omega^0(M) \simeq \mathbb{R}$ (because there are only n differentials available, so that an $(n+1)$ -form must vanish by virtue of (5.4).

This said, de Rham cohomology is defined as the following data (of finite dimensional \mathbb{R} -vector (sub)spaces) attached to $\Omega^\bullet(M)$:

$$(5.11) \quad H_{dR}^\nu(M) := \frac{\ker(\Omega^\nu(M) \xrightarrow{d} \Omega^{\nu+1}(M))}{\text{im}(\Omega^{\nu-1}(M) \xrightarrow{d} \Omega^\nu(M))}$$

or, more compactly, if we write d^ν for $\Omega^\nu(M) \xrightarrow{d} \Omega^{\nu+1}(M)$,

$$(5.12) \quad H_{dR}^\nu(M) = \frac{\ker(d^\nu)}{\text{im}(d^{\nu-1})}$$

In view of the finite length of $\Omega^\bullet(M)$ it follows that if $\nu < 0$ or $\nu > n = \dim(M)$, then $H_{dR}^\nu(M) = 0$.

Of course, the elements of $\ker(d^p)$ are, by definition, the closed p -forms, while the elements of $\text{im}(d^{p-1}) \subset H_{dR}^p(M)$ are the exact p -forms. *A propos*, the fact that $\text{im}(d^{p-1}) \subset \ker(d^p)$ is of course just a restatement of (5.7). Going in the other direction, if M is a real manifold, then the Poincaré Lemma asserts that on any contractible domain in M every closed p -form is also exact, i.e. the preceding containment is an equality: $\text{im}(d^{p-1}) = \ker(d^p)$, whence $H_{dR}^\nu(M) = 0$. This immediately leads to the observation that nonvanishing de Rham cohomology $H_{dR}^\nu(M)$, with $1 \leq \nu \leq \dim(M)$, must be a measure of the local geometric singularity of M . This was of some importance in Section 3, in the discussion of quantization.

Before we get to physics, however, we need a reformulation of de Rham cohomology in more flexible terms, specifically *vis à vis* the interplay between local and global data. This is provided by the theory of sheaves on topological spaces and the attendant sheaf cohomology.

Given any topological space X , e.g. a differentiable manifold ($X = M$), a sheaf over X is any topological space \mathcal{F} equipped with a projection mapping $\pi: \mathcal{F} \rightarrow X$ that is in fact a local homeomorphism: every point $x \in X$ possesses an open neighborhood U_x (with $x \in U_x$) such that the restricted mapping $\pi^{-1}|_{U_x}$ is a homeomorphism from U_x onto its image in \mathcal{F} . It is standard fare ([22], [12]) that this amounts to realizing \mathcal{F} as $\coprod_{x \in X} \mathcal{F}_x$, where \mathcal{F}_x is just the fibre $\pi^{-1}(x)$ of \mathcal{F} over x . And then the topology on \mathcal{F} is just the topology induced by that of X *via* π and π^{-1} . Beyond this there is the hugely important complementary perspective that realizes sheaves as contravariant functors on X 's topology, with additional structure, of course. Specifically, one realizes X 's topology as the category $\mathfrak{Top}(X)$ whose objects are the open sets $U \subset X$ of X 's topology, and whose morphisms are

the inclusions $U \subset V$ (for both U and V open in X); then the sheaf \mathcal{F} is realized as the contravariant functor

$$(5.13) \quad \mathcal{F} : \mathfrak{Top}(X) \rightarrow \mathfrak{Ab}$$

where the target category \mathfrak{Ab} is the category of abelian groups (with group homomorphisms as morphisms), such that (*qua* morphisms and contravariance) the image of an inclusion $U \subset V$ under the functor \mathcal{F} is a group homomorphism ("restriction") from the abelian group $\mathcal{F}(V)$ to the abelian group $\mathcal{F}(U)$, written $\varrho_U^V : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ subject to the rule that if $W \subset U \subset V$ in $\mathfrak{Top}(X)$ then $\varrho_W^V = \varrho_W^U \circ \varrho_U^V$. In addition a sheaf (as opposed to a mere presheaf, defined by the preceding requirements) must satisfy two more axioms: say that $U \in \mathfrak{Top}(X)$ is covered by open sets, i.e., $U = \cup_\alpha U_\alpha$; then, first, the condition that for each index α there exists a so-called section $\sigma_\alpha \in \mathcal{F}(U_\alpha)$ for which we have that, for all α, β , $\varrho_{U_\alpha \cap U_\beta}^{U_\alpha}(\sigma_\alpha) = \varrho_{U_\alpha \cap U_\beta}^{U_\beta}(\sigma_\beta)$, implies the existence of a section $\sigma \in \mathcal{F}(U)$ for which $\varrho_{U_\alpha}^U(\sigma) = \sigma_\alpha$ for all indices α ; and, second, if, with $\sigma \in \mathcal{F}(U)$, we have that $\varrho_{U_\alpha}^U(\sigma) = 0$ for every index α , then in fact the section σ vanishes on all of U , which means simply that as an element of the abelian group $\mathcal{F}(U)$, $\sigma = 0$. This latter way of phrasing things is part and parcel of the fact that the two foregoing characterizations of \mathcal{F} as a sheaf are equivalent.

Indeed, write $\Gamma(U, \mathcal{F})$ for the set of continuous sections of the topological space \mathcal{F} over the base topological space X , i.e.

$$(5.14) \quad \Gamma(U, \mathcal{F}) = \{\sigma : U \xrightarrow{cts} \mathcal{F} \mid \pi \circ \sigma = id|_U\},$$

so that $\Gamma(U, \mathcal{F})$ is an abelian group in its own right. Next, define the stalk of \mathcal{F} , regarded as a functor (as immediately above), over the point $x \in X$, to be

$$(5.15) \quad \lim_{\substack{\rightarrow \\ x \in U}} \mathcal{F}(U) \stackrel{def}{=} \cup_{x \in U} \mathcal{F}(U) / \sim$$

where the equivalence relation \sim is defined by the rule: given $\sigma, \tau \in \cup_{x \in U} \mathcal{F}(U)$, we have $\sigma \sim \tau$ if and only if, given that $\sigma \in \mathcal{F}(U)$ and $\tau \in \mathcal{F}(V)$ (with $x \in U, V$ as indicated), there exists an open $W \subset U \cap V$ such that $\varrho_W^U(\sigma) = \varrho_W^V(\tau)$. One shows (e.g. [22]) that this stalk is nothing else than our earlier fibre \mathcal{F}_x and that, in point of fact, the abelian groups $\mathcal{F}(U)$ and $\Gamma(U, \mathcal{F})$ can be identified:

$$(5.16) \quad \mathcal{F}(U) \approx \Gamma(U, \mathcal{F})$$

Accordingly, the phrase "section of \mathcal{F} over U " can be used ambiguously, or *avec un peu d'abus de langage*, as above, without penalty. To boot, we are at liberty to refer to \mathcal{F}_x as both the stalk and the fibre of \mathcal{F} over x .

Now, with the notion of the sheaf \mathcal{F} over X characterized in two equivalent ways we are in the fortunate position of being able to define the cohomology of the topological space X with values in \mathcal{F} in two rather natural fashions. First, with \mathcal{F} also being a topological space it is natural to develop $H^\bullet(X, \mathcal{F})$, i.e. the collective data $H^\nu(X, \mathcal{F})$, $\nu \in \mathbb{Z}$, of abelian groups (possibly with additional structure: we will eventually be working with vector spaces), by means of a variant of the classical Čech process; this is in fact the original approach taken by Serre in [20]. For our purposes it is far more advantageous, however, to employ Grothendieck's development of sheaf cohomology (cf. [10], [12]) by means of derived functors and injective and acyclic resolutions; see [13] for additional background in this connection. Although it requires a bit of work to prove this, for the spaces we are concerned with, namely smooth manifolds, the two indicated cohomology theories are equivalent.

First some generalities. Given any abelian category \mathfrak{A} (meaning that \mathfrak{A} admits injective, or one-one, morphisms, as well as surjective, or onto, morphisms) we can form short exact sequences which are, by definition, diagrams in \mathfrak{A} of the form

$$(5.17) \quad 0 \rightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \rightarrow 0,$$

which is to say, diagrams of concatenated morphisms of \mathfrak{A} , such that f (resp. g) is injective (resp. surjective), and $\text{im}(f) = \text{ker}(g)$, this last condition conveying the meaning of exactness at the "vertex" A . In general, given a concatenation, or sequence, of any number, finite or infinite, of morphisms of \mathfrak{A} , the sequence is exact if and only if it is exact at each vertex.

If \mathfrak{B} is a second abelian category (of course, we allow $\mathfrak{A} = \mathfrak{B}$), a covariant functor $T : \mathfrak{A} \rightarrow \mathfrak{B}$ is left exact if, given any short exact sequence in \mathfrak{A} of the form (4.17), the sequence

$$(5.18) \quad 0 \rightarrow T(A') \xrightarrow{T(f)} T(A) \xrightarrow{T(g)} T(A'')$$

is exact in \mathfrak{B} . Thus, while $T(f)$ is still required to be one-one, $T(g)$ need no longer be onto.

Next, an object I in \mathfrak{A} is said to be injective if, given any injective mapping $0 \rightarrow A' \xrightarrow{f} A$ and any morphism $A' \xrightarrow{\iota} I$, there exists a morphism $A \xrightarrow{\varpi} I$ yielding the factorization $\iota = \varpi \circ f$; we say that \mathfrak{A} has "enough injectives" if every object A in \mathfrak{A} can be fitted into a long exact sequence as follows:

$$(5.19) \quad 0 \rightarrow A \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \dots \xrightarrow{d^{\nu-1}} I^\nu \xrightarrow{d^\nu} I^{\nu+1} \xrightarrow{d^{\nu+1}} \dots$$

so that we have, for all ν , $d^\nu \circ d^{\nu-1} = 0$; the mapping ε is called an augmentation, and we often abbreviate this long exact sequence as

$$(5.20) \quad 0 \rightarrow A \xrightarrow{\varepsilon} I^\bullet$$

and this sequence is called an injective resolution of A . With all this in place, the usual cohomology of the "deleted" chain complex

$$(5.21) \quad 0 \rightarrow T(I^0) \xrightarrow{T(d^0)} T(I^1) \xrightarrow{T(d^1)} T(I^2) \xrightarrow{T(d^2)} \dots \xrightarrow{T(d^{\nu-1})} T(I^\nu) \xrightarrow{T(d^\nu)} T(I^{\nu+1}) \xrightarrow{T(d^{\nu+1})} \dots,$$

(or briefly just $T(I^\bullet)$), being the abelian group data

$$(5.22) \quad H^\nu(T(I^\bullet)) := \ker(T(d^\nu)) / \text{im}(T(d^{\nu-1})),$$

is defined to be the right derived functor data for $T(A)$ as an object in \mathfrak{B} , written as $R^\nu T(A)$, $\nu \in \mathbb{Z}$, or, briefly, $R^\bullet T(A)$, of course. In other words,

$$(5.23) \quad R^\nu T(A) = H^\nu(T(I^\bullet))$$

Cohomology is designed to be homotopy invariant, so, given that injective resolutions of the same object are (chain) homotopic, any of them will do the job; see [13] and [9] in this connection. Furthermore, we can require (loc. cit.) that the resolution in question be acyclic with respect to the functor T . By definition, an object \mathcal{A} in \mathfrak{A} is T -acyclic if its own cohomology vanishes in degrees ≥ 1 , and it is a standard result in homological algebra (cf. [4], [9]) that if we have an injective resolution $0 \rightarrow A \xrightarrow{\varepsilon} I^\bullet$ for A as well as a T -acyclic resolution $0 \rightarrow A \xrightarrow{\delta} \mathcal{A}^\bullet$ for the same object, then the application of T yields chain homotopic images: there is a natural chain homotopy from $0 \rightarrow T(A) \xrightarrow{T(\varepsilon)} T(I^\bullet)$ to $0 \rightarrow T(A) \xrightarrow{T(\delta)} T(\mathcal{A}^\bullet)$. Because chain homotopic cochains yield the same cohomology (loc. cit.), we get that

$$(5.24) \quad R^\nu T(A) = H^\nu(T(\mathcal{A}^\bullet))$$

Consequently, we can (and will) simply agree that as a matter of course, we choose T -acyclic resolutions to compute the right derived functors of T .

At this stage we can bring this machinery to bear on the situation of a sheaf \mathcal{F} on a topological space X . The main player in the game is the so-called global sections functor, $\Gamma(X, _)$, mapping sheaves over X to abelian groups. It is a standard result in sheaf theory (cf. [12], [22]) that the global sections functor is left exact, therefore, as per the preceding we can form the indicated right derived functors $R^\nu \Gamma(X, _)$ which, by definition, provide our cohomology: the cohomology of X with values in the sheaf \mathcal{F} is given by

$$(5.25) \quad H^\bullet(X, \mathcal{F}) := R^\bullet\Gamma(X, \mathcal{F})$$

By our earlier remarks, for the spaces we are working with, this gives the same group theoretic data (up to isomorphism) as what the aforementioned Čech process gives.

This agreement of cohomology theories holds true, then, in particular for the smooth manifold M for which we already have in place the de Rham cohomology $H_{dR}^\bullet(M)$ as given by (5.12). Is there a way to realize this cohomology as a sheaf cohomology? In other words, is there a sheaf on M , say \mathcal{G} , with the property that $H_{dR}^\nu(M) \approx H^\nu(M, \mathcal{G})$, for all ν ? It turns out that nothing less than Grothendieck's famous proof of the de Rham isomorphism theorem comes into play here, in a particularly elegant manner.

The de Rham theorem asserts that $H_{dR}^\bullet(M)$ is essentially the only cohomology on M in the sense that, for example, $H_{dR}^\bullet(M)$ must agree with the usual cohomologies one builds on a topological space like M , e.g., singular or Čech cohomology (which are *eo ipso* isomorphic to each other). So the claim is that $H_{dR}^\bullet(M) \approx H_{\check{C}ech}^\bullet(M, \mathbb{R})$, taking our cohomology with real coefficients. Grothendieck's way of proving this (in [11]) is as follows (we use Chern's rendering of the argument in [5]): first note that the constant sheaf \mathbb{R} on X (gotten by requiring that for all $x \in M$, $\mathcal{F}_x = \mathbb{R}$) admits a very special $\Gamma(M, _)$ -acyclic resolution, namely.

$$(5.26) \quad 0 \rightarrow \mathbb{R} \xrightarrow{\varepsilon} \Omega^0(M)^\dagger \xrightarrow{d^0} \Omega^1(M)^\dagger \xrightarrow{d^1} \Omega^2(M)^\dagger \xrightarrow{d^2} \dots \xrightarrow{d^{p-1}} \Omega^p(M)^\dagger \xrightarrow{d^p} \Omega^{p+1}(M)^\dagger \xrightarrow{d^{p+1}} \dots$$

where, generally, $\Omega^p(M)^\dagger$ is the sheaf (!) of germs of smooth p -forms on M . Grothendieck establishes that each such sheaf is "fine", meaning that given any locally finite open cover $\{U_\alpha\}_\alpha$ of M , there exists a set of sheaf endomorphisms $\varphi_\alpha : \Omega^p(M)^\dagger \rightarrow \Omega^p(M)^\dagger$ with the property that $\sum_\alpha \varphi_\alpha = id$ and if $x \notin U_\alpha$, then $\varphi_\alpha(\Omega^p(M)^\dagger_x) = 0$, and observes that fine sheaves are in fact $\Gamma(M, _)$ -acyclic. But then, in accord with the above discussion, the cohomology of M with coefficients in the constant sheaf \mathbb{R} is realized as $R^\bullet\Gamma(M, \mathbb{R})$, i.e., as the data given by the derived functors $R^p\Gamma(M, \mathbb{R}) = H^p(\Omega^\bullet(M)^\dagger)$, given (5.23). However, $R^p\Gamma(M, \mathbb{R})$ can readily be identified with $H_{\check{C}ech}^p(M, \mathbb{R})$, and it follows from the characterization (5.25) that $H^p(\Omega^\bullet(M)^\dagger) \approx H_{dR}^p(\Omega^\bullet(M)^\dagger) = H_{dR}^\bullet(M)$, briefly. But then, since, *qua* sheaf cohomologies, $R^p\Gamma(M, \mathbb{R}) \approx H_{\check{C}ech}^p(M, \mathbb{R})$, it follows that, for all p , $H_{dR}^p(M) \approx H_{\check{C}ech}^p(M, \mathbb{R})$, or, simply put, $H_{dR}^\bullet(M) \approx H_{\check{C}ech}^\bullet(M, \mathbb{R})$, as required.

By the way, the point of contact between Grothendieck's remarkably short proof of this deep theorem and other proofs, e.g., that given by Weil in [23], is the observation that for each degree p , the sheaf $\Omega^p(M)^\dagger$ is fine. In a way this is an encapsulation of the geometry at the center of the argument, rendered in particular topological and, given the nature of germs of p -forms, analytic terms.

We note, too, that, as we already hinted above, the usual formulation of the de Rham theorem involves other than Čech cohomology: its most frequent (albeit entirely equivalent) phrasing involves singular cohomology ([21], [23]), and brings out explicitly that the final isomorphism, $H_{dR}^p(M) \xrightarrow{\sim} H_{sing}^p(M, \mathbb{R})$, is induced by

the mapping taking any p -form ω to the mapping $\gamma \mapsto \int_{\gamma} \omega$, from the class of p -cycles γ to \mathbb{R} . The upshot is that since this isomorphism is surjective, we can find, for any pre-assigned set of periods, a closed p -form and an attendant collection of p -cycles such that the indicated integrations of the form yield these pre-assigned periods. It is this feature of de Rham's theorem that figures most prominently in what came before, regarding quantization.

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